Condensation phenomena of conserved-mass aggregation model on weighted complex networks

Sungchul Kwon, Sooyeon Yoon and Yup Kim
Department of Physics and Research Institute for Basic Sciences, Kyung Hee University, Seoul 130-701, Korea
(Dated: March 27, 2008)

We investigate the condensation phase transitions of conserved-mass aggregation (CA) model on weighted scale-free networks (WSFNs). In WSFNs, the weight w_{ij} is assigned to the link between the nodes i and j. We consider the symmetric weight given as $w_{ij} = (k_i k_j)^\alpha$. In CA model, the mass m_i on the randomly chosen node i diffuses to a linked neighbor of i,j, with the rate T_{ji} or an unit mass chips off from the node i to j with the rate ωT_{ji}. The hopping probability T_{ji} is given as $T_{ji} = w_{ji}/\sum_{<l>} w_{li}$, where the sum runs over the linked neighbors of the node i. On the WSFNs, we numerically show that a certain critical α_c exists below which CA model undergoes the same type of the condensation transitions as those of CA model on regular lattices. However for $\alpha \geq \alpha_c$, the condensation always occurs for any density ρ and ω. We analytically find $\alpha_c = (\gamma - 3)/2$ on the WSFN with the degree exponent γ. To obtain α_c, we analytically derive the scaling behavior of the stationary distribution P_k^∞ of finding a walker at nodes with degree k, and the probability $D(k)$ of finding two walkers simultaneously at the same node with degree k. We find $P_k^\infty \sim k^{\alpha+1-\gamma}$ and $D(k) \sim k^{2(\alpha+1)-\gamma}$ respectively. With P_k^∞, we also show analytically and numerically that the average mass $m(k)$ on a node with degree k scales as $k^{\alpha+1}$ without any jumps at the maximal degree of the network for any ρ as in the SFNs with $\alpha = 0$.

PACS numbers: 05.70.Fh,05.40.-a,89.75.Da,89.75.Hc

I. INTRODUCTION

A wide variety of mass transport systems ranging from traffic flow to polymer gels [1, 2, 3, 4, 5, 6, 7, 8] exhibit nonequilibrium condensation phenomena. These include basic microscopic dynamics ubiquitous in nature such as aggregation, fragmentation and diffusion. The nonequilibrium steady states of these systems are classified into two types of phases, so-called fluid phase and condensed phase. A finite fraction of total particles condenses on a single site in the condensed phase. In the fluid phase, the particle number on each site fluctuates around the density of total particles (ρ) without the condensation. As the rates of these processes vary, the condensation phase transitions between the two phases may take place at a certain critical density ρ_c.

One of the simplest mass transport models exhibiting the condensation transitions is a conserved-mass aggregation (CA) model [9, 10, 11, 12, 13]. CA model evolves via diffusion, chipping and aggregation upon contact which arise in a variety of phenomena such as polymer gels [4], the formation of colloidal suspensions [5], river networks [6, 7] and clouds [8]. In one-dimensional CA model, the mass m_i of site i moves either to site $i-1$ or to site $i+1$ with unit rate, and then $m_i \to 0$ and $m_{i\pm1} \to m_{i\pm1} + m_i$. With rate ω, unit mass chips off from site i and moves to one of the nearest neighboring sites; $m_i \to m_i - 1$ and $m_{i\pm1} \to m_{i\pm1} + 1$. The generalization to higher dimensions is straightforward. As total masses are conserved, the conserved density ρ and the rate ω determine the phase of CA model. The $\omega = \infty$ case corresponds to the well-known zero range process (ZRP) with a constant hopping rate [14, 15, 16].

The existence of the condensation transitions in CA model depends on the symmetry of movement, the constraints of diffusion rate and the underlying network structure [10, 11, 12, 13]. In the symmetric CA (SCA) model [9, 10] in which diffusion and chipping direction are unbiased, the condensation transitions take place at a certain ρ_c. The steady state properties of SCA model is exactly described by the mean field theory [10]. The single site mass distribution $P(m)$ was shown to undergo phase transitions on regular lattices [9]. For a fixed ω, as ρ is varied across the critical density $\rho_c(\omega)$, the behavior of $P(m)$ for large m was found to be [9]

$$P(m) \sim \begin{cases} e^{-m/m^*} & \rho < \rho_c(\omega), \\ m^{-\tau} & \rho = \rho_c(\omega), \\ m^{-\tau} + \text{infinite} \quad \text{aggregate} & \rho > \rho_c(\omega). \end{cases} \quad (1)$$

Mean field theory predicts $\rho_c(\omega) = \sqrt{\omega + 1} - 1$ and $\tau = 5/2$ [9, 10].

Recently, CA model on unweighted scale-free networks (SFNs) with degree distribution $P(k) \sim k^{-\gamma}$ was studied to investigate the effect of underlying network structure on the condensation transitions [13]. We call networks with equal weight on all links unweighted networks. On unweighted SFNs, the same type of the condensation transitions as those of SCA in regular lattice take place for $\gamma > 3$. However for $\gamma \leq 3$, the condensation always occurs for any density $\rho (> 0)$. It was shown that the existence of the transitions is directly related to the diffusive capture process on unweighted SFNs [13, 25].

On the other hand, most real-world networks exhibit not only a heterogeneous distribution of degree, but also heterogeneous distribution of weights [17, 18, 19]. Weights assigned on links characterize the interaction

strengths between nodes. There have been various attempts to understand the underlying mechanism and scale-free behaviors of empirically observed weighted networks [20]. Also there have been attempts to understand the effect of heterogeneous weights on various dynamics such as synchronization, dynamics of random walks, transport and percolation, and condensation of zero-range process [21, 22, 23]. These studies showed that dynamical properties are modified and exhibit non-trivial dependence on the strength of weight. In this paper, as the generalization of our study on CA model on complex networks, we investigate the effect of both heterogeneous degree and weight on the condensation phenomena of CA model on weighted networks.

The weight w_{ji} represents the weight to a link from the node i to j. In general, the strength s_i of the node i scales with the degree k_i as $s_i \sim k_i^{\alpha}$. The exponent α varies with network structures [19, 20]. Thus it is natural to take the weight w_{ji} as $w_{ji} \sim s_i s_j \sim (k_i k_j)^{\alpha}$.

In this paper, we study the condensation transitions of CA on the WSFNs with degree distribution $P(k) \sim k^{-\gamma}$ and the symmetric weight $w_{ji} = (k_i k_j)^{\alpha}$. As in one dimension, the diffusion of the whole masses and the fragmentation of unit mass occur with the unit rate and the rate ω, respectively. In addition, masses move from a node i to j with hopping rate proportional to $w_{ji}/\sum_j w_{ji}$. We found that a certain critical α_c exists below which the condensation transitions take place. However for $\alpha \geq \alpha_c$, the condensation always occurs for any density $\rho > 0$. To find α_c as a function of the degree exponent γ, one needs the steady state distribution P_k^{∞} of finding a walker at nodes with degree k on the WSFNs. P_k^{∞} gives the capture probability $D(k)$ with which two walkers meet at a node with degree k. We analytically derived P_k and $D(k)$, and finally obtained $\alpha_c = (\gamma - 3)/2$.

This paper is organized as follows. In Sec. II, we discuss the condensation transitions of CA model on the WSFNs. To verify the existence α_c, we investigate the steady state property of a single walker and the diffusive capture process on the WSFNs in Sec. III and IV. We discuss the behavior of an average mass $m(k)$ of a node with degree k in Sec. V and finally summarize our results in Sec. VI.

II. CA MODEL ON WSFNS WITH SYMMETRIC WEIGHTS

We consider CA model on WSFNs with the weight w_{ij} from node j to i defined as $w_{ij} = (k_i k_j)^{\alpha}$. For the construction of WSFN, we first construct an unweighted static SFN with N nodes and K links [26]. The degree k_i of a node i is defined as the number of its links connected to other nodes. The average degree of a node $< k >$ is given as $< k >= 2K/N$. The degree distribution $P(k)$ of SFNs is a power-law distribution $P(k) \sim k^{-\gamma}$. In the static model [26], it is desired to use large $< k >$ to construct fully connected networks. In simulations, we use $< k >= 4$. After then, we assign a weight $w_{ij} = (k_i k_j)^{\alpha}$ to the link between node i and j. Thus the hopping probability of masses from node i to an i's linked neighbor j is $T_{ji} = k_i^{\alpha} k_j^{\alpha} / \sum_{<m>} k_m^{\alpha} k_i^{\alpha} = k_j^{\alpha} / \sum_{<m>} k_m^{\alpha}$. $\sum_{<m>}$ denotes the sum over the linked neighbors of node i.

Each node has an integer number of particles, and the mass on a node is defined as the number of particles on the node. Initially M particles are randomly distributed on N nodes with given conserved density $\rho = M/N$. Next a node i is chosen at random and one of the following events occurs:

(i) Diffusion : With the unit rate, the whole mass m_i of node i moves to one of the linked neighbors j with probability T_{ji}. Then the aggregation takes place; $m_i \to 0$ and $m_j \to m_j + m_i$.

(ii) Chipping : With the rate ω, unit mass moves to a linked neighbor j with the probability T_{ji}, and then the aggregation takes place, i.e. $m_i \to m_i - 1$, $m_j \to m_j + 1$.

The $\omega = \infty$ case corresponds to ZRP with constant chipping rate on WSFNs [16].

We perform Monte Carlo simulations with random initial mass distribution on the WSFNs with $\gamma = 2.7$ and 4.0. We set $\omega = 1$ and the network size $N = 10^5$ with $< k >= 4$. We measure the single node mass distribution $P(m)$ in the steady states.

In Fig. 1, we plot $P(m)$ for $\gamma = 4.0$ with two different α, $\alpha = 0.05$ and 1.0. $P(m)$ exhibits quite different behavior according to the value of α. For $\alpha = 0.05$ (Fig. 1(a)), $P(m)$ decays exponentially without aggregates for sufficiently low density $\rho = 0.15$. On the other hand, for sufficiently high density, $\rho = 3.0$, an aggregate forms with the power-law decaying background mass distribution. It means that the condensation transition takes place at a certain critical density $\rho_c (> 0)$. Hence $P(m)$ follows Eq. (1). Since in unweighted SFNs, i.e $\alpha = 0$, the condensation phase transitions take place for $\gamma > 3$ [13], one may expect the condensation transitions for very small α. Based on the following steps, we estimate ρ_c and the exponent τ.

In the condensed phase, the total density ρ is written as $\rho = \rho_c + \rho_{\infty}$, where ρ_{∞} is the density of an aggregate. Since ρ is given as $\rho = \int_1^{\infty} mP(m)dm$, one can estimate ρ_c from $\rho_c = \int_1^{m_o} mP(m)dm$, where the upper bound m_o is the cut-off mass at which the background distribution terminates. Using this method, we estimate $\rho_c = 0.218$. We estimate the exponent τ from the scaling plot $m^{\tau} P(m)$ using $P(m)$ of $\rho = 3.0$ (Inset of Fig.1(a)). Since the background distribution does not change for $\rho \geq \rho_c$, we use $P(m)$ of $m \leq m_o$ for the scaling plot. We estimate $\tau = 2.38(5)$.

On the other hand, for $\alpha = 1.0$ (Fig.1(b)), $P(m)$ shows the complete different behavior. The condensation takes place with an exponentially decaying background distribution for both sufficiently low and high density, $\rho = 0.1$ and 3.0. Therefore we conclude that the condensation always occurs for any nonzero density so a system is always in the condensed phase without any transitions for $\alpha = 1.0$. The two different behaviors of $P(m)$ for

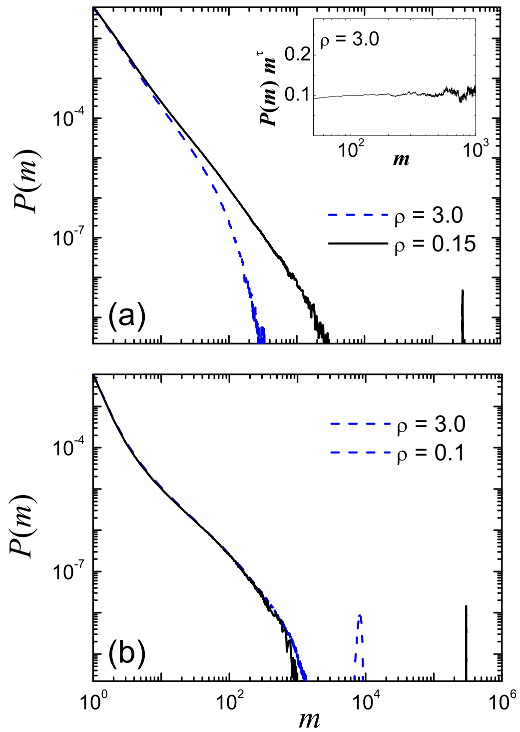

FIG. 1: (Color online) The plot of $P(m)$ for $\gamma = 4.0$ with $\alpha = 0.05$ (a) and $\alpha = 1.0$ (b). The inset of (a) shows the scaling plot $m^\tau P(m)$ with $\tau = 2.38$ when $\rho = 3.0$.

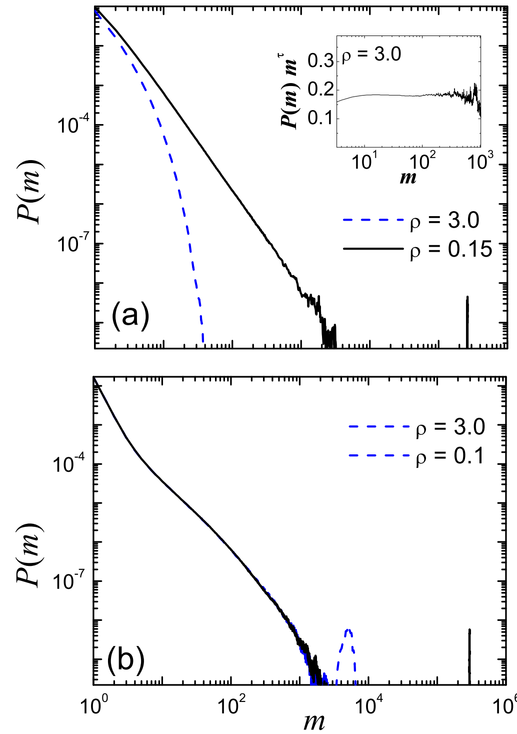

FIG. 2: (Color online) The plot of $P(m)$ for $\gamma = 2.7$ with $\alpha = -1.0$ (a) and $\alpha = -0.05$ (b). The inset of (a) shows the scaling plot $m^\tau P(m)$ with $\tau = 2.46$ when $\rho = 3.0$.

$\alpha = 0.05$ and 1.0 indicate that a crossover α (α_c) should exist in the range $0.05 < \alpha < 1.0$ for $\gamma = 4.0$. A system undergoes the condensation transition for $\alpha < \alpha_c$, while the condensation always occurs without the transition for $\alpha \geq \alpha_c$.

Similarly, for $\gamma = 2.7$, $P(m)$ exhibits the same different behavior according to the value of α. The difference from the $\gamma = 4.0$ case is that the condensation transitions are observed for a negative α. We observe the condensation transitions for $\alpha = -1.0$ (Fig. 2(a)). With the same method used in the $\gamma = 4.0$ case, we estimate $\rho_c = 0.4$ and $\tau = 2.46(5)$ respectively. However, for $\alpha = -0.05$, the condensation is observed even for very low density $\rho = 0.1$, which means that a system is always in the condensed phase for $\alpha = -0.05$ (Fig. 2(b)). Therefore, the crossover α_c also exists for $\gamma = 2.7$, but its value is negative unlike the $\gamma = 4.0$ case. Together with the results of $\gamma = 4.0$, we conclude that the crossover α_c exists for any $\gamma (> 2)$ and α_c varies with γ. In what follows, we discuss the existence of α_c and next the condensation nature for $\alpha < \alpha_c$.

First, the condensation phenomena of CA model on WSFNs is similar to that on unweighted SFNs. On unweighted SFNs, the condensation transitions exist for $\gamma > 3$, while the condensation always occurs for $\gamma \leq 3$ [13]. Hence the crossover γ is $\gamma_c = 3$. Intriguingly, it was shown that the existence of transitions is determined by the survival probability of a diffusing prey chased by a diffusing predator, so-called the lamb-lion problem [27]. The reason is as what follows.

In the limit $\rho \to 0$, let us assume only an infinite aggregate exists. With the rate ω, the unit mass is chipped off and moves around with the unit rate. If the chipped mass meets again with the infinite aggregate within a finite time interval, then the infinite aggregate is stable against the chipping process. On the other hand, if the chipped mass and the infinite aggregate does not meet again within a finite time interval, then the infinite aggregate would disappear by repeated chipping processes. Therefore, the stability of the infinite aggregate is physically related to the capture process in which a diffusing lion (infinite aggregate) chases a diffusing lamb (chipped mass). For the unweighted SFNs with $\gamma \leq 3$, it was shown that the survival probability $S(t)$ of a lamb decays exponentially with finite life time $\langle T \rangle_\infty$ [13, 25]. However, for $\gamma > 3$, $S(t)$ is finite in the thermodynamic limit. The behavior of $S(t)$ implies that the condensation transition exist for $\gamma > 3$ due to the stable fluid phase in the limit $\rho \to 0$, but only the condensation exist for $\gamma \leq 3$. As a result, the asymptotic behavior of the survival prob-

ability of a lamb in the lamb-lion capture process determines the existence of the condensation transitions on unweighted SFNs.

Similarly, on the WSFNs, the existence of the condensation transitions is also expected to depend on the survival probability $S(t)$ of a lamb. To see this, let us consider two limits, $\alpha \to +\infty$ and $-\infty$ for a given γ. In the limit $\alpha \to +\infty$, a walker always moves to a node with the larger degree. Once a walker reach the hub node with the maximal degree, the walker is trapped at the hub node forever. As a result, a lion always captures a lamb at the hub node within a finite time interval. Hence, $S(t)$ should decay exponentially with a finite life time. On the other hand, in the limit $\alpha \to -\infty$, a walker is forced to reach nodes with the minimal degree. Due to the inhomogeneous structure, the nodes with the minial degree are connected by nodes with larger degree. Hence, a walker cannot escape from one of the nodes with the minimum degree in this limit. It means that a lion cannot always capture a lamb at some other node, so that $S(t)$ is finite.

From the behavior of $S(t)$ in the two opposite limits, there should be a crossover α_c. $S(t)$ is finite for $\alpha < \alpha_c$ and decays to zero for $\alpha \geq \alpha_c$. For the condensation phenomena, one expects no condensation transitions ($\rho_c = 0$) for $\alpha \geq \alpha_c$ due to finite life time of a lamb. Instead, the condensation always occurs. On the other hand, the condensation transitions occur for $\alpha < \alpha_c$. We analytically find $\alpha_c = (\gamma - 3)/2$ for a given γ in Sec. IV. From $\alpha_c = (\gamma - 3)/2$, one reads $\alpha_c = -0.15$ for $\gamma = 2.7$ and $\alpha_c = 0.5$ for $\gamma = 4$ respectively. Our simulation results for $\gamma = 4$ and 2.7 confirm the existence of α_c and also the sign of α_c for each γ.

Next, we discuss the critical behavior of CA model on WSFNs. The CA model on any dimensional regular lattice and unweighted SFNs with $\gamma > 3$ is well described by mean-field theory [10, 13]. On WSFNs, interestingly, the transitions take place even for $\gamma < 3$, which means that the transition nature is not affected by the inhomogeneity of network structure. Since α_c diverges for $\gamma \to \infty$, the critical behavior of CA model on SFNs with $\alpha < \alpha_c$ should be the same as that on random networks where α, i.e. weight, does not have no special meaning due to the uniform degree distribution. As a result, one expects the mean-field critical behavior of SCA model in regular lattice. Our numerical estimates of τ, $\tau = 2.38(5)$ for $\gamma = 2.7$ and $\tau = 2.46(5)$ for 4.0, well agree with the mean-field value $\tau = 5/2$. Therefore, we conclude that the critical behavior of CA model for $\alpha < \alpha_c$ on the WSFNs belongs to the universality class of SCA model in regular lattice.

In summary, for a fixed γ, there is a crossover weight exponent α_c. CA model undergoes the same type of condensation transitions as those of SCA model in regular lattice for $\alpha < \alpha_c$, while the condensation always takes place for nonzero density for $\alpha \geq \alpha_c$. To find α_c as a function of the degree exponent γ, one needs the steady state distribution P_k^∞ of finding a walker at nodes with degree k on the WSFNs. In the next section, we derive P_k^∞ on the WSFNs. In Sec. IV, we study lamb-lion capture process on the WSFNs and finally find α_c using P_k^∞.

III. WALKS ON WSFNS WITH SYMMETRIC WEIGHTS

We consider a single walker on weighted networks with the weight w_{ij}. The connectivity of the network is represented by the adjacency matrix $\mathbf{A}$ whose element $A_{ij} = 1$ if there is a link from a node j to i. Otherwise, $A_{ij} = 0$. We set $A_{ii} = 0$ conventionally. The degree k_i of a node i is given as $k_i = \sum_j A_{ji}$. Since we consider weighted networks with weight w_{ij}, we define the weighted adjacency matrix $\widetilde{\mathbf{A}}$ as $\widetilde{A}_{ij} = w_{ij} A_{ij}$.

The motion of a walker on the weighted networks defined by the matrix $\widetilde{\mathbf{A}}$ is a stochastic process in the discrete time. We derive the stationary distribution P_i^∞ of a walker being at node i following the method of Ref. [24]. To set up the equation, we define the transition probability as follows. A walker at node i at time t selects one of its k_i linked nodes with hopping probability T_{ji}. Then, at time $t+1$, the walker moves to the selected node. The hopping probability T_{ji} from node i to j is then given as $T_{ji} = \widetilde{A}_{ji}/\widetilde{K}_i$, where $\widetilde{K}_i = \sum_j \widetilde{A}_{ji}$ is the strength of node i. As an initial condition, assume that the walker starts at the node q at time $t = 0$. Then the recurrence relation of the transition probability P_{iq} of finding the walker at node i at time t is

$$P_{iq}(t+1) = \sum_l T_{il} P_{lq}(t) \quad . \tag{2}$$

Then the transition probability $P_{iq}(t)$ is written by iterating as

$$P_{iq}(t) = \sum_{l_1, \cdots, l_{t-1}} T_{il_{t-1}} \cdots T_{l_2 l_1} T_{l_1 q} \quad . \tag{3}$$

For a symmetric $\widetilde{\mathbf{A}}$ with $\widetilde{A}_{ij} = \widetilde{A}_{ji}$, one finds $\widetilde{K}_q P_{iq}(t) = \widetilde{K}_i P_{qi}(t)$ by comparing P_{qi} and P_{iq}. In the stationary state, the probability P_i^∞ of finding a walker at node i should be independent of initial starting nodes, which gives $\widetilde{K}_i P_q^\infty = \widetilde{K}_q P_i^\infty$. Summing up over q, one finds

$$P_i^\infty = \widetilde{K}_i / \mathcal{N} \quad , \tag{4}$$

where $\mathcal{N} = \sum_{q=1}^N \widetilde{K}_q = \sum_{q=1}^N \sum_{m=1}^N \widetilde{A}_{mq}$. In weighted networks with symmetric weights, P_i^∞ is proportional to the strength of node i, i.e. the sum of the weights of the nearest neighboring nodes. The same result was found in the recent study on the dynamics of random walks on growing weighted networks [22].

In this paper, we consider the symmetric weight w_{ij},

$$w_{ij} = (k_i k_j)^\alpha \quad . \tag{5}$$

For the weight (5), P_i^∞ is not given as a simple form. Hence it is better to handle the distribution P_k^∞ of finding a walker at nodes with degree k. Using Eq. (4), one can see that

$$P_k^\infty = \sum_{i=1}^{N} P_i^\infty \delta_{k_i k} = \frac{1}{\mathcal{N}} \sum_{i=1}^{N}\sum_{j=1}^{N} A_{ji}(k_i k_j)^\alpha \delta_{k_i,k} \tag{6}$$

To express the sum in Eq. (6) in terms of degree k, we arrange the sum as follows. Only terms with $k_i = k$ contributes nontrivially to the sum $\sum_i$ and thus $NP(k)$ nodes with the degree k in a network have the nontrivial contributions to the sum. The node with the degree k has k linked neighbors whose degree ranges from 1 to the maximal degree of the network k_{max}. Hence, the number of nontrivial terms in the sum $\sum_{i=1}^N \sum_{j=1}^N$ is $NP(k)k$, which can be arranged in the order of increasing degree. Then, the double sum of Eq. (6) is written as $NP(k)k^{\alpha+1}(g(1)1 + g(2)2^\alpha + \cdots + g(k_{max})k_{max}^\alpha)$, where $g(k')$ is the degree distribution of the node involved in such $NP(k)k$ terms. For large N, we approximate $g(k')$ to $P(k')$. Then P_k^∞ is approximately given as

$$\begin{aligned} P_k^\infty &= \frac{N}{\mathcal{N}} P(k) k^{\alpha+1} \int_{k_0}^{k_{max}} P(k') k'^\alpha dk' \\ &= P(k)k^{\alpha+1} / \int_{k_0}^{k_{max}} P(k') k'^{\alpha+1}\, dk' \ . \end{aligned} \tag{7}$$

On SFNs with degree distribution $P(k) \sim k^{-\gamma}$, the integral in the second line is finite for $\alpha < \gamma - 2$. Hence we finally obtain P_k^∞ on WSFNs as

$$P_k^\infty \sim k^\sigma \ \ , \ \ \sigma = \alpha + 1 - \gamma \ . \tag{8}$$

The exponent σ varies with α and γ, and also changes its sign. For $\alpha = \gamma - 1$, i.e. $\sigma = 0$, P_k^∞ is independent of degree k so a walker does not feel the inhomogeneity of the underlying network structure. While a walker performs biased walks to nodes with the larger degree for $\sigma > 0$, the direction of the bias is reversed for $\sigma < 0$. Since the exponent α is a free parameter, one can controls the direction of the bias for a given γ.

To check the scaling relation (8), we perform Monte Carlo simulations on the WSFNs with $N = 10^5$ and the average degree $< k >= 4$. In the steady states, we measure P_k^∞ for various α up to 2.6 for $\gamma = 2.7$ and 3.0 for $\gamma = 3.3$. Fig. 3 shows the plot of P_k^∞ against k for several values of α. As shown, P_k^∞ scales in power-law with k. The inset in each panel shows the plot of σ against α. The simulation results agree well with the analytical prediction (8).

IV. CAPTURE PROCESS ON WSFNS WITH SYMMETRIC WEIGHTS

In this section, we consider the capture process or the lamb-lion problem on WSFNs with the symmetric weights (5). A lamb and a lion initially locate separately on randomly selected two nodes. Then the probability $D(k)$ of finding two walkers at the same node with degree k at the same time is proportional to $(P_k^\infty)^2$. From Eq. (8), one gets

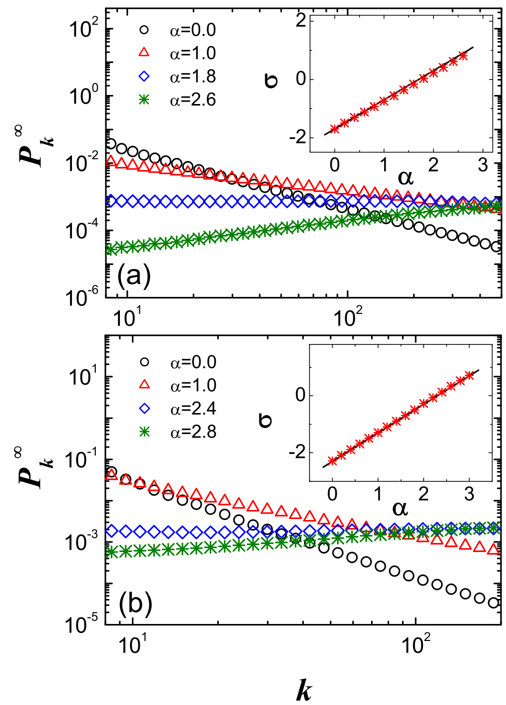

FIG. 3: (Color online) The plot of P_k^∞ and σ for $\gamma = 2.7$ (a) and $\gamma = 3.3$ (b). Insets show the relation (8) (solid line) and numerical estimates of σ (symbols).

$$D(k) = (P_k^\infty)^2 / NP(k) \sim k^\nu \tag{9}$$

with

$$\nu = 2(\alpha + 1) - \gamma \ . \tag{10}$$

Then the probability D of finding two walkers on the same node with any degree is given as

$$D = \int_{k_0}^{k_{max}} D(k) dk \sim \int_{k_0}^{k_{max}} k^{2(\alpha+1)-\gamma} dk \ . \tag{11}$$

Since the upper bound k_{max} diverges with N, the integral $\int_{k_0}^{k_{max}} k^{2(\alpha+1)-\gamma} dk$ diverges for $\alpha \geq (\gamma - 3)/2$. Hence there exists a crossover value α_c given as

$$\alpha_c = (\gamma - 3)/2 \ . \tag{12}$$

For $\alpha < \alpha_c$, the lamb survives indefinitely with a finite probability. However, for $\alpha \geq \alpha_c$, the lion captures the lamb with the unit probability. To check the scaling relation (10), we measure $D(k)$ on the WSFNs with $N = 10^5$

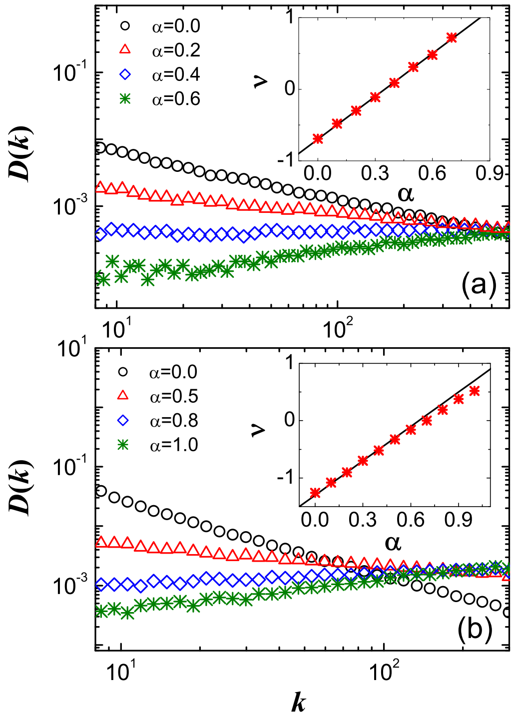

FIG. 4: (Color online) The plot of $D(k)$ and ν for $\gamma = 2.7$ (a) and $\gamma = 3.3$ (b). Insets show the relation of (10) (solid line) and numerical estimates of ν (symbols).

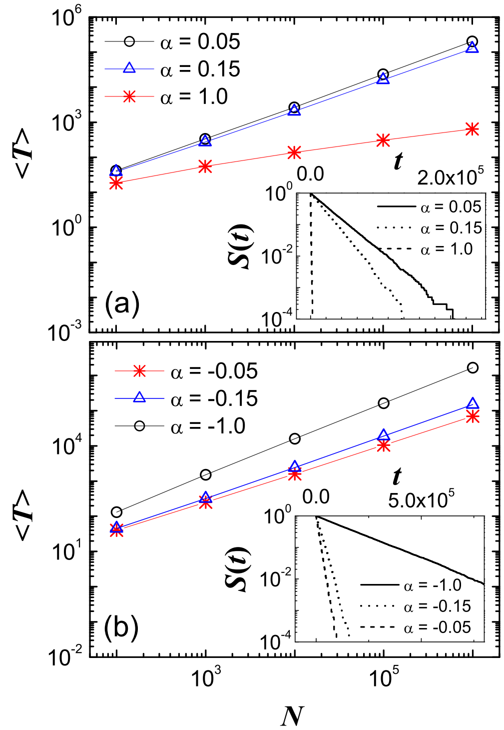

FIG. 5: (Color online) The average lifetime $\langle T\rangle$ of a lamb for $\gamma = 3.3$ (a) and $\gamma = 2.7$ (b). The solid line is a guide to the eye. Insets show the semi-logarithmic plots of $S(t)$ for $N = 10^5$. From top to bottom, each line corresponds to the $S(t)$ of $\alpha < \alpha_c$, α_c and $\alpha > \alpha_c$ respectively.

and $< k >= 4$. For 10^5 trials, we count the number $n(k)$ of capture events on nodes with the degree k. We obtain $D(k)$ to divide $n(k)$ by total trials(10^5). Fig. 4 shows the plot of $D(k)$ for several values of α, which scales well with k in power-law. As shown in the insets of Fig. 4, numerical estimates for ν satisfy the relation (10) very well.

To verify the existence of α_c by another method, we now consider the survival probability $S(t)$ of a lamb. $S(t)$ always satisfies $S(t) = S_\infty\ e^{-t/\tau}$ on random and scale-free networks due to the small world nature [13, 25]. As $S(t) = S_\infty\ e^{-t/\tau}$ in SFNs with any γ, we are interested in the average life time $\langle T\rangle$ of a lamb rather than $S(t)$ itself. From $\langle T\rangle = \int_0^\infty [-dS(t)/dt]\ dt$ and $S(t) = S_\infty\ e^{-t/\tau}$, we have $\langle T\rangle \sim \tau$. Hence $\langle T\rangle$ is infinite for $\alpha < \alpha_c$ and finite for $\alpha \geq \alpha_c$ in the limit $N \to \infty$. However, for the finite-sized networks, a lamb is eventually captured within N time steps for any α. For $\alpha < \alpha_c$, the maximum life time should be the order of N to guarantee the finite survival probability in the limit $N \to \infty$. Hence $\langle T\rangle$ is expected to scale as $\langle T\rangle \sim N$ for $\alpha < \alpha_c$. We measure $\langle T\rangle$ on WSFNs of $\gamma = 2.7$ and 3.3 with network size N up to 10^6. From Eq. (12), one reads $\alpha_c = -0.15$ for $\gamma = 2.7$ and $\alpha_c = 0.15$ for $\gamma = 3.3$.

In Fig. 5, we plot $\langle T\rangle$ against N. As shown in each inset, $S(t)$ exponentially decays for any α. For $\gamma = 3.3$(Fig. 5(a)), $\langle T\rangle$ increases with N as N^ϕ with $\phi = 0.94(1)$ for $\alpha = 0.05$ ($< \alpha_c$) and $\phi = 0.90(1)$ for $\alpha = \alpha_c(= 0.15)$. We estimate ϕ by measuring successive slopes from the log-log data in Fig. 5(a). For $\alpha = 1.0$ ($> \alpha_c$), $\langle T\rangle$ tends to saturate to the asymptotic value $\langle T\rangle_\infty$ with decreasing successive slopes. The exponent ϕ of $\alpha = 0.05$ is close to the expected value $\phi = 1$. For $\alpha = \alpha_c(= 0.15)$, $\langle T\rangle$ seems to diverge with $\phi = 0.9$. However, since $\langle T\rangle$ of $\alpha = 1.0$ already tends to saturate, it is expected that $\langle T\rangle$ of $\alpha \geq \alpha_c$ would saturate to a finite value in the network with $N > N_c(\alpha)$, where $N_c(\alpha)$ is the characteristic size for given α. For example, for $\alpha = 1.0$, $\langle T\rangle$ does not get into the saturation region even after $N = 10^6$, which implies $N_c(1.0) > 10^6$ for $\alpha = 1.0$. Since N_c should increase as $\alpha \to \alpha_c$, it is empirically impossible to see the saturation of $\langle T\rangle$ via simulations. Therefore, the initial slope ϕ at α_c may have no special meaning as that of $\alpha > \alpha_c$. The same behavior for $\langle T\rangle$ was observed for $\alpha = 0$ case [13], where $\langle T\rangle$ initially algebraically increases with continuously varying $\phi(< 1)$ as $\gamma \to 3$ from below.

For $\gamma = 2.7$ (Fig. 5(b)), we estimate $\phi = 1.00(2)$ for $\alpha = -1.0$ ($< \alpha_c$) as expected. However, for $\alpha = -0.05$ ($> \alpha_c$), $\langle T\rangle$ algebraically increases with $\phi = 0.82(1)$. Since for $\alpha = 0$ [13], N_c is already larger than 10^6 for $\gamma = 2.75$,

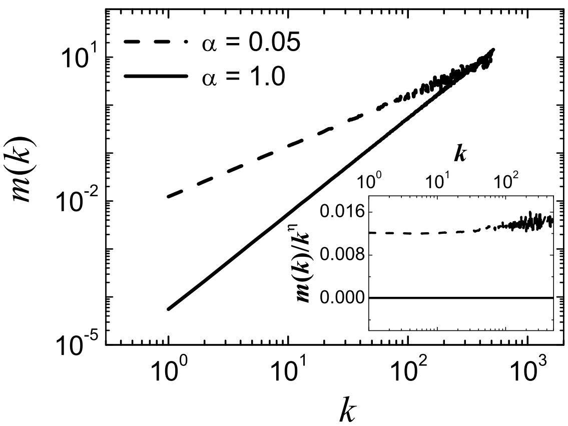

FIG. 6: The plot of $m(k)$ for $\gamma = 3.3$. The solid and the dashed line correspond to $\alpha = 1.0$ and 0.05 respectively. The inset shows the scaling plot $m(k)k^{-\eta}$ with $\eta = 1.06$ for $\alpha = 0.05$ (dashed line) and $\eta = 1.95$ for $\alpha = 1.0$ (solid line).

it is difficult to see the saturation of $\langle T\rangle$. For $\alpha = \alpha_c$, we estimate $\phi = 0.90(1)$. As in $\gamma = 3.3$, the initial slope for $\alpha \geq \alpha_c$ has no special meaning. Based on our numerical results, we are convinced that $\langle T\rangle$ approaches a finite value $\langle T\rangle_\infty$ for $\alpha \geq \alpha_c$ and becomes infinite for $\alpha < \alpha_c$ in the limit $N \to \infty$. Hence in the limit $N \to \infty$, we have

$$\lim_{N\to\infty} S(N,t) = \begin{cases} S_o e^{-t/\tau_\infty} & (\alpha \geq \alpha_c) \\ S_\infty & (\alpha < \alpha_c) \end{cases} . \qquad (13)$$

with the characteristic time $\tau_\infty \sim \langle T\rangle_\infty$.

V. AVERAGE MASS OF A NODE WITH DEGREE k

Another interesting quantity in condensation phenomena on networks is the average mass $m(k)$ at a node with degree k in the steady state [13, 15, 16]. In ZRP with chipping rate $u(m) \sim m^\delta$, the complete condensation takes place for $\delta < \delta_c$, where $\delta_c = 1/(\gamma-1)$ for unweighted SFNs [15] and $(\alpha+1)/(\gamma-1)$ for WSFNs with the weight (5) [16]. For $\delta < \delta_c$, $m(k)$ increases as $k^{\alpha+1}$ for $k < k_c$, and as $k^{(\alpha+1)/\delta}$ for $k \geq k_c$ on the WSFNs. Especially for $\delta = 0$, $m(k)$ increases $k^{\alpha+1}$ until $k < k_{max}$ and jumps to the value $m_{hub} \approx \rho N$ at k_{max}. Hence the condensation takes place at the node with k_{max} degree in ZRP.

The recent study on CA model on unweighted SFNs showed that $m(k)$ linearly increases up to k_{max} without the jump at k_{max} unlike in ZRP with constant chipping rate [13]. The linearity of $m(k)$ comes from the fact that all masses can diffuse. The mass m_{hub} formed on the node with the degree k_{max} can diffuse throughout network to make the steady sate distribution P_i^∞. By taking average over all nodes, the m_{hub} soaks into the average mass $m(k)$ unlike in ZRP where all samples have m_{hub} at k_{max}. The linearity of $m(k)$ on unweighted SFNs results from $P_k^\infty \sim kP(k)$ [13]. Therefore, from $P_k^\infty \sim k^{\alpha+1}P(k)$ on the WSFNs, we expect $m(k) \sim k^{\alpha+1}$ up to k_{hub}. To see this explicitly, we derive the relation $m_k \sim k^{\alpha+1}$ as follows.

We consider the average total mass $M(k)$ of nodes with degree k defined as

$$M(k) = \sum_{m=0}^{\infty} mP_\infty(m,k) \ , \qquad (14)$$

where $P_\infty(m,k)$ is the probability of finding a walker with mass m at nodes with degree k in the steady state. Since the mass distribution $P(m)$ in the steady state is independent of k, we write $P_\infty(m,k) = P(m)P_k^\infty$. From (7) and (14), one reads

$$M(k) \simeq k^{\alpha+1}P(k)\sum_{m=0}^{\infty} mP(m) \ , \qquad (15)$$

where we drop the normalization constant of P_k^∞. Since the number of nodes with degree k is $NP(k)$, $m(k)$ is given as

$$m(k) = \frac{M(k)}{NP(k)} \sim k^{\alpha+1} \ . \qquad (16)$$

To confirm the scaling behavior of $m(k)$, we measure $m(k)$ in the condensed phase on the WSFNs of $\gamma = 3.3$ and $N = 10^5$. In Fig. 6, we plot $m(k)$ against k for $\alpha = 0.05$ and 1.0. With $\omega = 1$, we set $\rho = 3.0$ which corresponds to the condensed phase for both α values. Assuming $m(k) \sim k^\eta$, one expects $\eta = 1.05$ for $\alpha = 0.05$ and 2.0 for $\alpha = 1.0$ respectively. From the scaling plot $m(k)/k^\eta$ (Inset of Fig. 6), we estimate $\eta = 1.06(2)$ for $\alpha = 0.05$ and $\eta = 1.95(5)$ for $\alpha = 1.0$ which agree well with the predictions.

VI. SUMMARY

In summary, we investigate the properties of conserved-mass aggregation (CA) model on weighted scale-free networks (WSFNs). In WSFNs, the weight w_{ij} is assigned to the link between node i and j. We consider the symmetric weight given as $w_{ij} = (k_ik_j)^\alpha$. In CA model, masses diffuse with unit rate and unit mass chips off from mass with rate ω. In addition, the hopping probability T_{ji} from node i to j is given as $T_{ji} = w_{ji}/\sum_{<m>} w_{mi}$.

On the WSFNs, a walker finally reaches the hub node with k_{max} degree for $\alpha \to \infty$, while it is trapped forever at nodes with the minimal degree for $\alpha \to -\infty$. In the lamb-lion capture process, it means that the lion captures

the lamb at the hub node within finite time interval for $\alpha \to \infty$. On the other hand, a lamb survives indefinitely with finite probability for $\alpha \to -\infty$, because the lion cannot escape from a node with the minimal degree to capture a lamb at some other node. In-between the two limits, one expects a crossover α_c below which the life time $\langle T \rangle$ of a lamb is infinite. However, for $\alpha \geq \alpha_c$, $\langle T \rangle$ is finite. The dependence of $\langle T \rangle$ on α is similar to that on unweighted SFNs of $\alpha = 0$ where $\langle T \rangle$ is infinite for $\gamma > 3$ and finite otherwise [13].

To verify the existence of α_c, we need the stationary distribution P_k^{∞} of finding a walker at nodes with degree k. From the equation for the transition probability $P_{ji}(t)$ to go from node i to j in t time steps, we analytically find $P_k^{\infty} \sim k^{\alpha+1-\gamma}$. Next, we consider the so-called lamb-lion capture process. With P_k^{∞}, we find the probability $D(k)$ of finding two walkers at the same node with degree k at the same time to scale $D(k) \sim k^{2(\alpha+1)-\gamma}$. Finally, integrating out $D(k)$, we find the death probability D of a lamb. A lamb survives indefinitely with the finite survival probability for $\alpha < \alpha_c$, while it is eventually captured by a lion for $\alpha \geq \alpha_c$. We analytically find $\alpha_c = (\gamma - 3)/2$. Therefore, in the limit $N \to \infty$, the life time $\langle T \rangle$ of a lamb is finite for $\alpha \geq \alpha_c$, while it is infinite for $\alpha < \alpha_c$. We numerically confirm the all analytical results.

The existence of the condensation transitions is known to depend on $\langle T \rangle$ of a lamb [13]. For $\alpha \geq \alpha_c$, $\langle T \rangle$ is finite so the condensation always occurs for any nonzero density. On the other hand, for $\alpha < \alpha_c$, the infinite $\langle T \rangle$ ensures the condensation transitions at a certain critical density ρ_c. For $\alpha \geq \alpha_c$, we numerically confirm that the condensation always takes place at very low density. We also numerically confirm that for $\alpha < \alpha_c$, CA model on the WSFNs undergoes the same type of the condensation transitions as those of SCA model in regular lattice.

Finally, we investigate the behavior of the average mass $m(k)$ of a node with degree k. In ZRP with constant chopping rate on networks [15, 16], $m(k)$ increases as $k^{\alpha+1}$, and jumps to the total mass of the system at k_{max}. However, in the SCA model on unweighted SFNs, it was shown that $m(k)$ linearly increases with k up to k_{max} without any jumps [13]. Furthermore the linearity of $m(k)$ is valid for any $\rho > 0$, which comes from the fact that the diffusion is only the relevant physical factor to decide the distribution $m(k)$. Similarly, on the WSFNs, we analytically find and numerically confirm that $m(k)$ algebraically increases as $k^{\alpha+1}$ for any $\rho > 0$ without any jumps.

Acknowledgments

This work was supported by the Korea Science and Engineering Foundation(KOSEF) grant funded by the Korea government(MOST) (No. R01-2007-000-10910-0) and by the Korea Research Foundation Grant funded by the Korean Government (MOEHRD, Basic Research Promotion Fund) (KRF-2007-313-C00279).

[1] M. R. Evans, Europhys. Lett. 36, 13 (1996).
[2] O. J. O'Loan, M. R. Evans, and M. E. Cates, Phys. Rev. E 58, 1404 (1998).
[3] D. van der Meer et al., J. Stat. Mech. Theor. Exp. 04, P04004 (2004); J. Torok, cond-mat/0407567.
[4] R. M. Ziff, J. Stat. Phys. 23, 241 (1980).
[5] W. H. White, J. Colloid Interface Sci. 87, 204 (1982).
[6] A. E. Scheidegger, Bull. I.A.S.H. 12, 15 (1967).
[7] A. Maritan, A. Rinaldo, R. Rigon, A. Giacometti, and I. R. Iturbe, Phys. Rev. E 53, 1510 (1996); M. Cieplak, A. Giacometti, A. Maritan, A. Rinaldo, I. R. Iturbe, and J. R. Banavar, J. Stat. Phys. 91, 1 (1998).
[8] S. K. Friedlander, Smoke, Dust and Haze (Wiley Interscience, New York, 1977).
[9] S. N. Majumdar, S. Krishnamurthy, and M. Barma, Phys. Rev. Lett. 81, 3691 (1998); S. N. Majumdar, S. Krishnamurthy, and M.Barma, J. Stat. Phys. 99, 1 (2000).
[10] R. Rajesh and S. N. Majumdar, Phys. Rev. E **63**, 036114 (2001).
[11] R. Rajesh and S. Krishnamurthy, Rhys. Rev. E 66, 046132(2002).
[12] R. Rajesh, D. Das, B. Chakraborty, and M. Barma, Phys. Rev. E **66**, 056104(2002).
[13] S. Kwon, S. Lee, and Yup Kim, Phys. Rev. E **73**, 056102 (2006).
[14] M. R. Evans, Braz.J.Phys. 30, 42 (2000); M. R. Evans and T. Hanney, J. Phys. A **38**, R195 (2005).
[15] J. D. Noh, G. M. Shim, and H. Lee, Phys. Rev. Lett. 94, 198701 (2005).
[16] M. Tang, Z. Liu, and J. Zhou, Phys. Rev. E **74**, 036101 (2006).
[17] S. L. Pimm, *Food Webs* (University of Chicago Press. Chicago. 2002). 2nd Ed.; A. E. Krause et al, Nature (London) **426**, 282 (2003).
[18] M. E. J. Newman, Phys. Rev. E **64**, 016132 (2001).
[19] A. Barrat,M. Barthélemy, R. Pastor-Santorras, and A. Vespignani, Proc. Natl. Acad. Sci. U.S.A. **101**, 3747 (2004).
[20] S. H. Yook, H. Jeong, A.-L. Barabasi, and Y. Tu, Phys. Rev. Lett. bf 85, 5835 (2001); A. Barrat, M. Barthelemy, and A. Vespignani, *ibid*, **92**, 228701 (2004); K.-I. Goh, B. Kahng, and D. Kim, Phys. Rev. E **72**, 017103 (2006).
[21] A. E. Motter, C. S. Zhou, and J. Kurths, Phys. Rev. E **71**, 016116 (2005).
[22] An-Cai Wu et al, Chin. Phys. Lett **24**, 557 (2007).
[23] G. Li et al, Phys. Rev. E **75**, 045103(R) (2007).
[24] J. D. Noh and H. Rieger, Phys. Rev. Lett. **92**, 118701 (2004).
[25] S. Lee, S. -H. Yook, and Yup Kim, Phys. Rev. E **74**, 046118 (2006).
[26] K.-I. Goh, B. Kahng, and D. Kim, Phys. Rev. Lett. **87**, 278701 (2001).
[27] P. L. Krapivsky and S.Redner, J. Phys. A 29, 5347 (1996).